MA 242 : Partial Differential Equations (August-December, 2018)

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Problem set 1

1. Find and sketch some sample characteristic curves of the PDE

$$(x+2)u_x + 2yu_y = 2u$$

in the $x-y$ plane. Write the ODE for u along a characteristic curve with x as parameter and then, solve the PDE with the initial condition $u(-1,y) = \sqrt{|y|}$.

2. Consider the PDE

$$xu_x + yu_y = 2u,\ x > 0, y > 0.$$

Plot the characteristic curves and then, solve the equation with the following initial conditions in the domain given above: (a) $u = 1$ on the hyperbola $xy = 1$; and (b) $u = 1$ on the circle $x^2 + y^2 = 1$.

Can you solve the equation, in general, if certain initial data is prescribed on the initial curve $y = e^x$? Justify with reasons.

3. Sketch the characteristic curve; the initial curve and solve the following problems

   (a) $xu_x + yu_y = ku, \quad x \in \mathbb{R},\ y \geq \alpha > 0; \quad u(x,\alpha) = F(x)$, where k, α are fixed and F is a given smooth function.

   (b) $(x+2)u_x + 2yu_y = \alpha u; \quad u(-1,y) = \sqrt{y}$.

   (c) $yu_x - xu_y = 0; \quad u(x,0) = x^2$.

   (d) $x^2u_x - y^2u_y = 0; \quad u(1,y) = F(y)$.

4. Find the characteristic curves of the following equations

   (a) $(x^2 - y^2 + 1)u_x + 2xyu_y = 0$.

   (b) $2xyu_x - (x^2 + y^2)u_y = 0$.

5. Solve the quasi-linear problem and verify transversality conditions.

(a) $uu_x + u_y = 0, \quad u(x,0) = x.$

(b) $uu_x + u_y = 1, \quad u(x,x) = x/2, x \in (0,1].$

6. Find and sketch the characteristic curves of $uu_x + u_y = 0$ with the following initial conditions

   (a) $u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

   (b) $u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$

   (c) $u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 1 \end{cases}$ and $u(x,0)$ is smooth and increasing.

7. Write down the characteristic ODE system for the equation $u_y = u_x^3$ and then, solve with the initial condition $2x^{3/2}$ on the $x-$ axis.

8. Find the integral surface of the equation $x(\frac{\partial u}{\partial x})^2 + y\frac{\partial u}{\partial y} = u$ passing through the line $y = 1,\ x + z = 0$.

9. Consider the equation $p^2 + q^2 = 1$ with initial condition $u(x,y) = 0$ on the line $x + y = 1$. Find two solutions using the method of characteristics.

10. Solve the following IVP:

    (a)
    $$u_t - \sqrt{u_{x_1}^2 + u_{x_2}^2} = 0,\ u(x_1, x_2, t_0) = \psi(x_1^2 + x_2^2),$$
    where $\psi' > 0$; here $(x_1, x_2) \in \mathbb{R}^2$ and $t > t_0$.

    (b)
    $$u_t + x\cos t u_x = 0,\ u(x,0) = \frac{1}{1+x^2}\ x \in \mathbb{R}, t > 0.$$

    (c)
    $$u_t + x^2 u_x = 0, u(x,0) = \phi(x),\ x \in \mathbb{R}, t > 0.$$

    (d)
    $$u_t + \frac{1}{1+|x|}u_x = 0, u(x,0) = \phi(x),\ x \in \mathbb{R}, t > 0.$$

    (e)
    $$u_t + (x+t)u_x + t(x+1)u = 0, u(x,0) = \phi(x),\ x \in \mathbb{R}, t > 0.$$

    (f)
    $$u_t + u^2 u_x = 0, u(x,0) = x,\ x \in \mathbb{R}, t > 0.$$